

Two Dimensional Gravity as a modified Yang-Mills Theory

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We study a deSitter/Anti-deSitter/Poincare Yang-Mills theory of gravity in d -space-time dimensions in an attempt to retain the best features of both general relativity and Yang-Mills theory: quadratic curvature, dimensionless coupling and background independence. We derive the equations of motion for Lie algebra valued scalars and show that in the geometric optics limit they traverse geodesics with respect to the Lorentzian geometry determined by the frame fields. Mixing between components appears to next to leading order in the WKB approximation. We then restrict to two space-time dimensions for simplicity, in which case the theory reduces to the well known Katanaev-Volovich model. We complete the Hamiltonian analysis of the vacuum theory and use it to prove a generalized Birkhoff theorem. There are two classes of solutions: with torsion and without torsion. The former are parametrized by two constants of motion, have event horizons for certain ranges of the parameters and a curvature singularity. The latter yield a unique solution, up to diffeomorphisms, that describes a space constant curvature.

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I. INTRODUCTION

General Relativity is often called a gauge theory of the gravitational field, but it is not a gauge theory of the Yang-Mills type. In the latter, the action functional $S_{YM}[A]$ is quadratic in the curvature of a connection A of principle bundle over the spacetime manifold (M_D, \mathbf{g}) , where \mathbf{g} is a *given* non-dynamical Lorentzian metric on M_D :

$$S_{YM}[A] = \frac{1}{8g_{YM}^2} \int_{M_D} d^D x \sqrt{-g} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha}^A F_{\nu\beta}^B h_{AB}. \quad (1)$$

In the above, g_{YM} is the gauge coupling constant. It has dimension $\text{Length}^{\frac{D}{2}-2}$, and hence is dimensionless in 4D; the $g^{\mu\nu}$ are the contravariant components of the metric tensor \mathbf{g} ; the $F_{\mu\alpha}^A$ are the components of the curvature of the

connection; the indices $A, B = 1, 2, \dots, n$ are in the adjoint representation of the gauge group; and finally h_{AB} are components of the Cartan-Killing metric of the group. It is the fact that g_{YM}^2 is dimensionless in 4-D which permits Yang-Mills gauge theories to be perturbatively renormalizable

By contrast, the Einstein-Hilbert action of General Relativity is linear in the curvature of the Christoffel connection of the Lorentzian metric. There is no background: the metric \mathbf{g} is dynamical. Moreover, the coupling constant in Einstein gravity has dimension Length^{-2} in four spacetime dimensions. It is this which has stalled progress in constructing quantum gravity starting from Einstein's theory.

It was perhaps Townsend [1] who first highlighted the fact that the gravitational constant has the dubious distinction of being the only dimensionful fundamental constant (the others being \hbar and c) that is tied to a specific dynamical theory. He therefore suggested that the gravitational constant G should somehow be linked directly to the structure of spacetime. This could be achieved by replacing the Poincare group as a potential local gauge symmetry of gravity by the deSitter group, which necessarily entails a dimensionful constant. With this as motivation, he proceeded to consider a Yang-Mills type Lagrangian for gravity with the deSitter group as gauge group.

Besides those of Townsend, there have in fact been many attempts to construct a Yang-Mills type gravitational theory. The first was by Weyl almost one hundred years ago, and the goal was to unify gravity with electromagnetism [2]. Work in the 70's and 80's, inspired by the work of Utiyama, Yang and Mills on non-Abelian gauge theories, constructed Yang-Mills type theories with gauge groups associated with gravity, for example the Poincare, DeSitter/anti DeSitter and Conformal groups [3]. More recently, J. T. Wheeler[4] and collaborators have worked on 4D Yang-Mill gravity, with the conformal group $\text{SO}(4,2)$ as the gauge group, while H.-Y. Guo[5] and his collaborators have tackled the de Sitter case.

In the first order formalism of Einstein gravity- the so-called Einstein-Cartan action- the equations of motion force the torsion to be zero. In Yang-Mills gravity this does not happen. Generically, the spacetime geometry has non-vanishing torsion as well as (quasi-)Riemannian curvature [3, 5]. The consequences of this for the viability of such theories is still an open question. We note here the result of [6] that torsion de-stabilizes anti-de Sitter 2+1 dimensional spacetime.

In spite of its quantum motivation, little progress has been made in quantizing Yang-Mills gravity. In fact, to date, there has been no canonical analysis of such theories, a necessary first step towards understanding the quantum theory. In this paper we begin to close this gap. After a general discussion that includes a discussion of the coupling to matter showing that, to leading order in the geometric optics limit, Higgs fields propagate along geodesics of the Lorentzian geometry, we will undertake the construction of the canonical form of a toy model of Yang-Mills gravity, wherein the spacetime is two dimensional, and the gauge group is the lineland version of de Sitter/anti-de Sitter/Poincare gravity, that is, $\text{SO}(2,1)/\text{SO}(1,2)/\text{ISO}(1,1)$. In this case, the Lagrangian reduces to a special case of the Katanaev-Volkov model[7], which was extensively studied in a somewhat different context in the 1990's[8]. We will solve the Hamiltonian equations of motion for the vacuum case, finding in the case of zero torsion that the solutions are equivalent to those of Jackiw-Teitelboim dilaton gravity [9].

II. GENERAL FORMALISM

A. Algebra and Action

In this section we outline the general procedure for constructing a gauge theory of gravity in a D -dimensional spacetime. We note here the record of such attempts sampled in [1–5]. The ‘kinematical’ gauge group associated with such a theory is one $\text{SO}(D,1)/\text{SO}(D-1,2)/\text{ISO}(D-1,1)$, corresponding to positive/negative/zero cosmological constant. The generators $J_A = (J_a, F_{ab})$ with $A = 0, 1, \dots, D$; $a, b = 0, 1, \dots, D-1$ obey

$$[J_a, J_b] = -2\eta_{DD}J_{ab}; \quad (2)$$

$$[J_a, J_{bc}] = -\frac{1}{2}(\eta_{ac}J_b - \eta_{ab}J_c); \quad (3)$$

$$[J_{ab}, J_{cd}] = -\frac{1}{2}(\eta_{ac}J_{bd} + \eta_{bd}J_{ac} - \eta_{bc}J_{ad} - \eta_{ad}J_{bc}), \quad (4)$$

where η_{ab} is the (D) -dimensional Minkowski metric. If $\eta_{DD} = 1, -1, 0$, then the gauge group is, respectively, $\text{SO}(D,1)/\text{SO}(D-1,2)/\text{ISO}(D-1,1)$.

The gauge potential is decomposed according to

$$A_\mu = \lambda e_\mu^a J_a + \omega_\mu^{ab} J_{ab}, \quad (5)$$

where the constant λ has dimension L^{-1} , so that with the vielbein e_μ^a dimensionless, the spin-connection ω_μ^{ab} and the gauge potential A_μ have dimension L^{-1} . The generator of translations, i.e. the 2-momentum, P_a has dimension L^{-1} and is related to the above via:

$$P_a = \lambda J_a \quad (6)$$

When working in terms of P_a , λ appears directly in the commutator algebra as opposed to the definition of the gauge potential. It is for this reason that Townsend[1] considered it to be a property of the spacetime structure, rather than a coupling constant.

Note that although it would be natural to identify λ with the dimensionally appropriate power of the g_{YM} , up to a dimensionless number of order unity, in order to keep things as general as possible we keep them distinct in what follows.

We note here that the structure constants for any of the DS/ADS/Poincare gauge groups have structure constants

$$\begin{aligned} f^{[cd]}_{ab} &= -2\eta_{DD}\delta_a^{[c}\delta_b^{d]}; \\ f^d_{a[bc]} &= -\delta_{[b}^d\eta_{c]a}; \\ f^{ef}_{[ab][cd]} &= -2\delta_{[a}^{[e}\eta_{b][d}\delta_{c]}^f]. \end{aligned} \quad (7)$$

The Cartan-Killing metrics on the gauge groups, defined by $h_{ij} := 2f^k_{il}f^l_{jk}$ are all of the form

$$h_{ab} = -2D\eta_{DD}\eta_{ab}; h_{[ab][cd]} = -D(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad}). \quad (8)$$

The field strength is defined as usual:

$$\begin{aligned} F &= \frac{1}{2}F_{\mu\nu}^{ij}dx^\mu \wedge dx^\nu := dA + \frac{1}{2}[A, A] \\ &= \lambda T^a J_a + \Omega^{ab} J_{ab}. \end{aligned} \quad (9)$$

Thus $[F] = L^{-2}$. The Lie algebra valued 2-forms Ω and T are respectively the ‘curvature plus volume element’ and torsion of the spin connection ω :

$$\Omega^{ab} := d\omega^{ab} + \omega^{ac} \wedge \omega_c^b - \lambda^2 \eta_{DD} e^a \wedge e^b; \quad (10)$$

$$T^a := de^a + \omega^a_b \wedge e^b. \quad (11)$$

In the above, indices $a, b, \dots = 0, 1, 2, D-2$ are raised and lowered by the Minkowski metric η_{ab} . and e.g. $F^a := \frac{1}{2}F_{ij}^a dx^i \wedge dx^j$. Thus $[T_{\mu\nu}^a] = L^{-1}$ and $[\Omega_{\mu\nu}^{ab}] = L^{-2}$. Most importantly, the ‘background metric’,

$$g_{\mu\nu} := \eta_{ab} e_\mu^a e_\nu^b, \quad (12)$$

is not fixed, but is subject to the dynamics determined by the equations of motion for the gauge field.

The action can be written explicitly in the form $S = S_{EH} + S_1$, where

$$S_{EH} := \frac{D\lambda^2}{2g_{YM}^2} \int d^D x \sqrt{-g} \left(R - \lambda^2 \eta_{DD} \frac{D(D-1)}{2} \right); \quad (13)$$

$$S_1 := -\frac{D}{4g_{YM}^2} \int d^D x \sqrt{-g} \left(\frac{K}{2} + \lambda^2 \eta_{DD} T_{a\mu\nu} T^{a\mu\nu} \right), \quad (14)$$

where $K := R^{ab\mu\nu} R_{ab\mu\nu}$.

Comparing the term S_{EH} to the usual Einstein-Hilbert action, we find that the Newton gravitational constant in D dimensions, G_D , (in units where the speed of light is one) is related to the gauge coupling constant G and the scale factor λ by

$$G_D = \frac{g_{YM}^2}{8\pi D\lambda^2}. \quad (15)$$

In general G_D has dimensions of L^{D-2} , so that it is dimensionless in 2 spacetime dimensions. We also remark again that $D = 4$ is also special in that g_{YM} is dimensionless.

To close this section, we consider the issue of the background metric \mathbf{g} . In the following, as in most of the literature on Yang-Mill gravity, the ‘background’ $g_{\mu\nu}$ will not really be a background, but is rather, dynamical, via $g_{\mu\nu} := \eta_{ab} e_\mu^a e_\nu^b$. One pays a price for this, however, in that the gauge transformations generated by the ‘translations’ J_a no longer preserve the action: some of the gauge symmetry is broken.

B. Adding Matter

Torsion theories are distinguished by the fact that the field content describes more than one kind of geometry. There is curvature associated with the Riemannian metric used to raise and lower indices, and there is also the Riemann-Cartan connection and associated curvature. When the torsion is non-zero, the metric compatible with the Riemann-Cartan connection is not the same as the Riemannian metric constructed out of the vierbeins/vielbeins. The only way to decide which geometry is relevant in a particular physical context is to look at matter couplings.

It is straightforward to add most forms of matter using the principle of minimal couple. Only spinors will couple directly to the torsion, whereas all other matter Lagrangians will just depend on $e^a{}_\mu$. Here we consider a Higgs-like scalar $\phi^A(x)$ that takes its values in the adjoint representation and couples to the vacuum action via $S = S_{YM} + S_{higgs}$, where

$$S_{higgs} = \int_{M_2} d^2x \sqrt{-g} g^{\mu\nu} h_{ij} D_\mu \phi^i D_\nu \phi^j. \quad (16)$$

Thus the matter field obeys the gauge covariant wave equation

$$D_\mu D^\mu \phi^i = 0. \quad (17)$$

In the geometric optics limit, a wave field has approximately constant amplitude, but varying phase. Thus for a Higgs type of matter we write

$$\phi^j = R^j e^{iS^j/\hbar} \quad (18)$$

Note that the Lie algebra index $j = 0, 1, 2$ is not summed over here, or subsequently.

Now the geometric optics limit has particles traveling with momenta $k_\mu^{(i)} = \partial_\mu S^i$ orthogonal to the constant surfaces $S^i(t, x) = \text{const.}$ Also, ∇_μ is the Lorentzian covariant derivative with respect to the background metric $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$. We assume that the amplitudes R^j are slowly varying compared to the phase. The wave equation $D^\mu D_\mu \phi^j = 0$ becomes, after dropping terms in $\partial_\mu R^j$ and keeping only terms of leading and subleading orders ($1/\hbar^2, 1/\hbar$, respectively)

$$0 = -\frac{1}{\hbar^2} e^{iS^j/\hbar} R^j k^{(j)\mu} k_\mu^{(j)} + \frac{i}{\hbar} \left(e^{iS^j/\hbar} R^j \nabla^\mu k_\mu^{(j)} + 2e^{iS^l/\hbar} f^j{}_{kl} A^{(k)\mu} k_\mu^{(l)} R^l \right). \quad (19)$$

Hence, to leading order

$$C^j := -R^j k^{(j)\mu} k_\mu^{(j)} = 0 \quad (20)$$

Note that this expression is real. The gauge covariant derivative of C^j reduces to the partial derivative. Thus, to leading order $k^{j\nu} \nabla_\mu k_\nu^j = 0$. Using the smoothness of the phase S^j in order to change the order of partial differentiation we find that $k^{j\nu} \nabla_\nu k_\mu^j = 0$. Thus to leading order the trajectories are null geodesics of the Lorentzian geometry compatible with the frame-field e_μ^i on spacetime.

The subleading terms are pure imaginary terms, and hence

$$e^{iS^j/\hbar} R^j \nabla^\mu k_\mu^{(j)} + 2e^{iS^l/\hbar} f^j{}_{kl} A^{(k)\mu} k_\mu^{(l)} R^l = 0 \quad (21)$$

The latter are more complicated because of the relative phase factor.

III. 1+1 DIMENSIONS: ACTION AND COVARIANT EQUATIONS OF MOTION

Things simplify quite a bit in 1+1 dimensions. The group is $SO(2, 1), SO(1, 2), ISO(1, 1)$ respectively, for $k := -\eta_{22} = -1, +1, 0$ with generators J_a, J , and algebra:

$$[J, J] = 0 \quad (22)$$

$$[J_a, J_b] = k \epsilon_{ab} J$$

$$[J, J_a] = \epsilon_a{}^b J_b \quad (23)$$

Note that J generates an Abelian one dimensional subalgebra.

$$A = \lambda e^a J_a + \omega J \quad (24)$$

As before we split the curvature into

$$F = \Omega J + \lambda T^a J_a \quad (25)$$

where $\Omega := d\omega + \frac{k}{2}\lambda^2 \epsilon_{ab} e^a \wedge e^b$ and $T^a := de^a - \epsilon^a_b \omega \wedge e^b$. That is:

$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + k\lambda^2 V_{\mu\nu} \quad (26)$$

$$T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu - \epsilon^a_b (\omega_\mu e^b_\nu - \omega_\nu e^b_\mu) \quad (27)$$

with $V_{\mu\nu} := \epsilon_{ab} e^a_\mu e^b_\nu$. To recover the expressions from the previous section, in an arbitrary number of dimensions, we replace $\omega = \frac{1}{2}\epsilon^{ab}\omega_{ab}$. Note that $V^{\mu\nu}V_{\mu\nu} = -2$ and $k := -\eta_{22}$. The cartan metric $h_{ij} = h_{ji}$ is defined to be:

$$h_{ij} := -2f_{ki}^l f_{lj}^k \quad (28)$$

with components:

$$h_{22} = -4 \quad (29)$$

$$h_{ab} = -4k\eta_{ab} \quad (30)$$

$$h_{a2} = 0 \quad (31)$$

The action Eq.(1) becomes:

$$S_{YM} = \frac{1}{4\lambda^2} \int d^2x \sqrt{-g} \left(-\tilde{R}^2 - k\lambda^2 T^a \eta_{ab} T^b + 2k^2 \lambda^4 + 2k\lambda^2 V^{\mu\nu} \tilde{R}_{\mu\nu} \right), \quad (32)$$

with

$$\tilde{R}_{\mu\nu} := \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \quad (33)$$

The last term in the action corresponds to the usual Einstein-Cartan term. In 2-dimensions it is a total divergence and will be dropped. Note that when $k = 0$, the above action reduces simply to a single term, namely the curvature-squared term.

The action (32) corresponds is of the same form as the Katanaev-Volovich model of 2-D gravity with torsion[7, 8], albeit with a specific ratio of coefficients determined by the gauge coupling parameter.

The equations of motion are the critical points of the action functional (32). That is, since the the spin-connection ω_μ and the frame-fields e^a_μ are functionally independent and

$$\delta S = \frac{1}{\lambda^2} \int d^2x \sqrt{-g} (W^\mu \delta \omega_\mu + E^{a\mu} \delta e_{a\mu}), \quad (34)$$

we have that a necessary condition for a critical point is that

$$W^\mu := -\nabla_\nu \tilde{R}^{\mu\nu} - C \epsilon_{ab} e^a_\nu T^{b\mu\nu} = 0; \quad (35)$$

$$E^{a\mu} := -C D_\nu T^{a\mu\nu} + e^a_\nu \tau^{\mu\nu} + \frac{C^2}{2} e^{a\mu} = 0. \quad (36)$$

where we have defined $C := k\lambda^2$. As well, the above spacetime tensor indices μ, ν, \dots are raised and lowered by the ‘background metric’ $g_{\mu\nu} := \eta_{ab} e^a_\mu e^b_\nu$. There are two covariant derivatives. The first, ∇_ν , is with respect to the background Lorentzian metric $g_{\mu\nu}$, while the second, D_ν is with respect to the spin-connection. That is

$$D_\nu T^{a\mu\nu} := \nabla_\nu T^{a\mu\nu} - \epsilon^a_b \omega_\nu T^{b\mu\nu}. \quad (37)$$

Finally, the tensor $\tau^{\mu\nu}$ is defined as

$$\begin{aligned} \tau^{\mu\nu} &:= \tilde{R}_\pi{}^\mu \tilde{R}^{\pi\nu} + C \eta_{ab} T^a{}_\pi{}^\mu T^{b\pi\nu} \\ &\quad - \frac{1}{4} g^{\mu\nu} (\tilde{R}^2 + C T^2), \end{aligned} \quad (38)$$

where $\tilde{R}^2 := \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu}$ and $T^2 := \eta_{ab} T^a{}_{\mu\nu} T^{b\mu\nu}$.

IV. HAMILTONIAN ANALYSIS

We parametrize the ‘background metric’ $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$, where the frame-field components are the J_a components of the gauge potential $A_\mu = \lambda e_\mu^a J_a + \omega_\mu J$. We write:

$$e^0 = ndt + p dx; \quad e^1 = q N^1 dt + q dx. \quad (39)$$

We note that the metric is:

$$g_{\mu\nu} = \begin{pmatrix} -(n^2 - q^2(N^1)^2) & -np + q^2 N^1 \\ -np + q^2 N^1 & q^2 - p^2 \end{pmatrix} \quad (40)$$

$$\sqrt{-g} = qN, \quad (41)$$

where $N := n - N^1 p$. Note also that

$$g^{\mu\nu} = \frac{1}{q^2 N^2} \begin{pmatrix} -(q^2 - p^2) & -np + q^2 N^1 \\ -np + q^2 N^1 & (n^2 - q^2(N^1)^2) \end{pmatrix} \quad (42)$$

Another potentially useful form of the metric is:

$$ds^2 = -\frac{q^2 N^2}{q^2 - p^2} dt^2 + (q^2 - p^2) \left(dx + \frac{q^2 N^1 - np}{q^2 - p^2} dt \right)^2 \quad (43)$$

As before, we define:

$$F = \partial_0 \omega_1 - \partial_1 \omega_0, \quad (44)$$

and

$$\begin{aligned} T^0 &= \dot{p} - n' + q(\omega_1 N^1 - \omega_0); \\ T^1 &= \dot{q} - (q N^1)' - \omega_0 p + \omega_1 n. \end{aligned} \quad (45)$$

The action is

$$S_{YM} = \frac{1}{2\lambda^2} \int d^2 x \left[\frac{1}{Nq} (F^2 + k\lambda^2 (T^1)^2 - k\lambda^2 (T^0)^2) + \lambda^4 Nq \right]. \quad (46)$$

Note that for the group ISO(1,1) (i.e. $k = 0$) only the F^2 term remains. In two dimensions this gives a rather trivial solution space so we henceforth consider only $k = \pm 1$.

The momenta canonically conjugate to $p, q, n, N^1, \omega_0, \omega_1$ are respectively

$$\Pi_p = -\frac{kT^0}{Nq}; \quad (47)$$

$$\Pi_q = \frac{kT^1}{Nq}; \quad (48)$$

$$\Pi_n = 0; \quad (49)$$

$$\Pi_1 = 0; \quad (50)$$

$$P_0 = 0; \quad (51)$$

$$P_1 = \frac{F}{\lambda^2 Nq}. \quad (52)$$

The total Hamiltonian density is ‘pure constraint’:

$$H = NH_s + N^1 D + \omega_0 M, \quad (53)$$

where the Hamiltonian constraint H_s is

$$H_s := \frac{q}{2} (-k\Pi_p^2 + k\Pi_q^2 + \lambda^2 P_1^2) - \frac{1}{4} q - D\Pi_p, \quad (54)$$

the diffeo constraint D is

$$D := -qD\Pi_q - pD\Pi_p, \quad (55)$$

where $D\Pi_q := \Pi'_q + \omega_1\Pi_p$ and $D\Pi_p := \Pi'_p + \omega_1\Pi_q$. Finally the ‘Gauss law constraint’ is

$$M := -P'_1 + q\Pi_p + p\Pi_q. \quad (56)$$

Note that above, $\Pi'_q := \partial_x \Pi_q$ is the spatial derivative.

The self-consistency of these constraints, that is $0 \approx \dot{H}_s(x) = [H_s(x), \int dy H(y)]$, etc., must be checked. We smear the constraints: $H_s[u] := \int dy u(y) H_s(y)$, etc., and find

$$[H_s[u], H_s[v]] = 0; \quad (57)$$

$$[H_s[u], M[v]] = \int dx \frac{uv}{q} (pH_s - D) \approx 0; \quad (58)$$

$$\begin{aligned} [H_s[u], D[v]] &= \int dx \left\{ \lambda^2 uvq P_1 M + \frac{uv\omega_1}{q} D - \frac{v}{q} (qu' + u\omega_1 p) H_s \right\} \\ &\approx 0; \end{aligned} \quad (59)$$

$$[D[u], M[v]] = 0; \quad (60)$$

and this is a strong equality.

$$[D[u], D[v]] = D[uv' - vu'] \approx 0. \quad (61)$$

And finally

$$[M[u], M[v]] = 0, \quad (62)$$

strongly.

We see that the constraint algebra closes, and the constraints are self-consistent.

The equations of motion are:

$$\dot{\omega}_1 = \lambda^2 NqP_1 + \omega'_0; \quad (63)$$

$$\dot{q} = kNq\Pi_q - N\omega_1 + (qN^1)' - N_1p\omega_1 + \omega_0p; \quad (64)$$

$$\dot{p} = -Nkq\Pi_p + N' + (N^1p)' - N^1q\omega_1 + \omega_0q; \quad (65)$$

$$\dot{P}_1 = N\Pi_q + N^1(q\Pi_p + p\Pi_q) \quad (66)$$

$$\dot{\Pi}_q = -\frac{k}{2}N\Omega + N^1D\Pi_q - \omega_0\Pi_p; \quad (67)$$

$$\dot{\Pi}_p = N^1D\Pi_p - \omega_0\Pi_q. \quad (68)$$

In the above we have defined:

$$\tilde{\Omega} := \frac{1}{2} (-k\Pi_p^2 + k\Pi_q^2 + \lambda^2 P_1^2 - \lambda^2) \quad (69)$$

V. SOLUTIONS

The gauge is fixed by

$$p \approx 0, \omega_1 \approx 0, Q := q - 1 \approx 0. \quad (70)$$

The consistency conditions for this choice are, respectively:

$$N' + \omega_0 - kN\Pi_p \approx 0; \quad (71)$$

$$\omega'_0 + kCNP_1 \approx 0; \quad (72)$$

$$(N^1)' + kN\Pi_q \approx 0. \quad (73)$$

The constraints reduce to

$$H_s = \frac{k}{2}\tilde{\Omega} - \Pi'_p \approx 0; \quad (74)$$

$$D = -\Pi'_q \approx 0; \quad (75)$$

$$M = \Pi_p - P'_1 \approx 0. \quad (76)$$

Now from Eq.(75) we have that $\Pi_q = \Pi_q(t)$ is an integration (spatial) constant. We use this and Eq.(76)(which allows us to replace Π_p by P'_1) in Eq.(74) to get the second order differential equation

$$P''_1 + \frac{k}{2}(P'_1)^2 - \frac{\lambda^2}{2}P_1^2 - B = 0, \quad (77)$$

where

$$B := \frac{k}{2}\Pi_q^2 - \frac{\lambda^2}{2}. \quad (78)$$

There are two classes of solutions to (77). This can be seen as follows. Define:

$$C_1 := e^{kP_1} [(P'_1)^2 - \lambda^4(kP_1 - 1)^2 + \Pi_q^2] \quad (79)$$

It is easy to verify that

$$C'_1 = \frac{1}{2}e^{kP_1} P'_1 \left[P''_1 + \frac{k}{2}(P'_1)^2 - \frac{\lambda^2}{2}P_1^2 - B \right] \quad (80)$$

Thus the solutions bifurcate into two classes:

$$P'_1 = 0 \rightarrow \frac{\lambda^2}{2}P_1^2 - B \quad (81)$$

$$P'_1 \neq 0 \rightarrow C_1 = C_1(t) \quad (82)$$

As we will see, the first condition requires that the torsion be zero. It leads to a solution-space of lower dimension. The second condition allows for non-zero torsion.

A. Torsion-less Solutions

This class of solutions of the Hamiltonian constraint has $P'_1 = 0$, and hence

$$P_1^2 + \frac{k\Pi_q^2}{\lambda^2} = 1. \quad (83)$$

In this case the Gauss Law constraint $M = 0$ implies $\Pi_p = 0$. If we now use these in the equation of motion (68) for $\dot{\Pi}_p$, we find that either $\omega_0 = 0$ or $\Pi_q = 0$. If we use the former in the consistency condition $0 = \dot{\omega}_1 = \omega'_0 + \lambda^2 NP_1$, then either $N = 0$, which leads to a degenerate geometry, or $P_1 = 0$. But $P_1 = 0$ implies $\dot{P}_1 = 0$, and hence the equation of motion for \dot{P}_1 implies $\Pi_q = 0$. Hence we must have that both Π_p and Π_q are zero; that is, the metric is torsion-free. Since $\Pi_q = 0$ we now find from (83) above that $P_1^2 = k^2 = 1$.

We now find that from the consistency conditions (71) and (72) that

$$N' + \omega_0 \approx 0; \quad (84)$$

$$\omega'_0 + k\lambda^2 N \approx 0; \quad (85)$$

that $N'' = k\lambda^2 N$ and $N^1 = N^1(t)$. The function N is then of the form $N_0(t) \sin \lambda(x - x_0(t))$, respectively $N_0(t) \sinh \lambda(x - x_0(t))$, as $k > 0$, respectively $k < 0$. In these expressions, $N_0(t), x_0(t)$ are integration constants. The metric is then of the form (with $k < 0$):

$$ds^2 = -(N_0(t) \sinh \lambda(x - x_0(t)))^2 dt^2 + (dx + N^1(t)dt)^2. \quad (86)$$

We have not completely fixed the coordinate invariance. One can choose:

$$dy = dx + N^1(t)dt \quad (87)$$

As well, the lapse can be set to one using the residual time reparameterization invariance so that the metric becomes:

$$ds^2 = -\sinh^2(\lambda(y - y_0(t)))dt^2 + dy^2. \quad (88)$$

where

$$y_0(t) := x_0(t) + \int dt N_1(t) \quad (89)$$

The remaining free function $y_0(t)$ is related to the fact that we have not completely fixed the gauge invariance. Indeed, the nontrivial consistency conditions, (60, 61), for the torsionless case, where $\Pi_p = 0, P_1 = \pm 1/\alpha$ boil down to (84) and (85), respectively, which can be written as a matrix equation

$$\phi' = A\phi \quad (90)$$

where $\phi = [N, \omega_0]^T$ and

$$A := \begin{pmatrix} 0 & -1 \\ -k\lambda^2 & 0 \end{pmatrix}$$

The system is preserved under linear transformations $\phi \rightarrow \bar{\phi} = L\phi$, where L is a 2x2 matrix:

$$L = \begin{pmatrix} b_1 & b_2 \\ k\lambda^2 b_2 & b_1 \end{pmatrix}$$

which has unit determinant if $b_1^2 - k\lambda^2 b_2^2 = 1$. Note that b_i are functions of t . This is an $O(1,1)$ transformation. We can use such a transformation to transform $\hat{x}_0(t)$ away. Indeed, such a transformation is given by $b_1(t) = -b_0(t)\lambda \cosh(\lambda y_0(t)), b_2(t) = b_0(t) \sinh(\lambda y_0(t))$. This gives $N \propto \sinh(\lambda y)$. Note that in this case $b_1^2 - k\lambda^2 b_2^2 = k\lambda^2 b_0^2(t)$. On the other hand, if we choose to transform so that $N \propto \cosh(\lambda y)$, then we would have different b_1, b_2 satisfying $b_1^2 - k\lambda^2 b_2^2 = -k\lambda^2 b_0^2(t)$. Thus only one of these transformations is continuously connected to the identity transformation.

Now consider the stationary metric

$$ds^2 = -n^2(z)d\tau^2 + q^2(z)dz^2. \quad (91)$$

For this to have constant curvature, that is, for $R = -2\lambda^2$, it is required that

$$qn'' - q'n' - k\lambda^2 q^3 n = 0. \quad (92)$$

If you solve this for $n(z)$ with $k = -1$ you get

$$n(z) = A \cosh(\lambda(\theta)) + B \sinh(\lambda(\theta)), \quad (93)$$

where in general the integration constants A, B can be τ -dependent. In the above, $\theta' = q(z)$. Now choose y as a new coordinate, so that $\int dz q(z) = y - y_0(\tau)$. The special case $A = 0$ is then

$$ds^2 = -\sinh^2(\lambda(y - y_0(t)))dt^2 + dy^2. \quad (94)$$

where we have scaled B away by a trivial coordinate transformation $dt = B(\tau)d\tau$.

The Ricci scalar for this metric is $R = -2\lambda^2$, a metric with constant negative curvature. For $k > 0$ we get the above, but with \sinh replaced by \sin . In this case $R = 2\lambda^2$, giving us a space of constant positive curvature.

This solution, as we have seen is torsionless, and of constant curvature. In fact, the solution has a flat Yang-Mills connection, that is, $F_{\mu\nu}^i = 0$. Furthermore, the expression $\tau^{\mu\nu}$, which is quadratic in the Yang-Mills curvature, satisfies $\tau^{\mu\nu} + \frac{\lambda^4}{2}g^{\mu\nu} = 0$.

B. Torsion-full Solution

One can verify that in the case $P'_1 \neq 0$, the following is the first integral of Eq.(77)

$$\begin{aligned} (P'_1)^2 &= [C_1 e^{-kP_1} + k\lambda^2(kP_1 - 1)^2 + \Pi_q^2] \\ &=: f^2(t, P_1) \end{aligned} \quad (95)$$

where $C_1(t)$ is an integration constant. We note also that as we have seen above, $\Pi'_q = 0$ and hence $\Pi_q = \Pi_q(t)$.

Now it is easy to see that Π_p is given as a function of P_1 from Eq.(76) by

$$\Pi_p = P'_1 = f(t, P_1) = \pm \frac{1}{k} [C_1 e^{-kP_1} + k\lambda^2(kP_1 - 1)^2 + \Pi_q^2]^{\frac{1}{2}}. \quad (96)$$

Differentiate (71) with respect to x . We write, for notational ease $r := P_1$. We get

$$N'' + \omega'_0 - k(Nf)' = 0. \quad (97)$$

We replace ω'_0 according to (72) and write $f' = \partial_x r f_r$ where $f_r = \partial_r f$ to get

$$f^2 N_{rr} + (f f_r - k f^2) N_r - k(f f_r + C r) N = 0. \quad (98)$$

The general solution is (according to MAPLE)

$$N(t, r) = f e^{kr} (B_1(t)g + B_2(t)), \quad (99)$$

where $B_1(t), B_2(t)$ are integration constants and $g(t, r)$ is given by $g_r = e^{-kr} f^{-3}$. We can now compute $\omega_0(t, r)$ from (71) to get

$$\omega_0(t, r) = -f_r N - \frac{B_1(t)}{f}. \quad (100)$$

From (73) we find

$$(N^1)_r = -k \Pi_q e^{kr} (B_1(t)g + B_2(t)). \quad (101)$$

Now consider the equations of motion for the time derivatives of the momenta:

$$\dot{r} = N \Pi_q + N^1 f; \quad (102)$$

$$\dot{\Pi}_q = -f(N f_r + \omega_0); \quad (103)$$

$$\dot{f} = N^1 f f_r - \omega_0 \Pi_q. \quad (104)$$

Consider the linear combination

$$f_r \dot{r} - \dot{f} = (N f_r + \omega_0) \Pi_q, \quad (105)$$

by (102) and (104). Use (103) on the right hand side to get

$$\Pi_q \dot{\Pi}_q + f(f_r \dot{r} - \dot{f}) = 0. \quad (106)$$

According to (95)

$$\frac{d}{dt} f^2 = \dot{r} (f^2)_r + \dot{C}_1(t) e^{-kr} + 2 \Pi_q \dot{\Pi}_q. \quad (107)$$

Thus using this in (106) we get $\dot{C}_1 = 0$ and hence C_1 is both a space and time constant.

From (103) we then get $\dot{\Pi}_q = B_1(t)$.

Consider again (104). After multiplying by f . The left hand side becomes, after using (95)

$$\frac{df^2}{dt} = 2k \Pi_q \dot{\Pi}_q. \quad (108)$$

Hence (104) becomes

$$2k\Pi_q\dot{\Pi}_q = (fN^1 + N\Pi_q)(f^2)_r + 2B_1\Pi_q, \quad (109)$$

so that after canceling the left side with the last term on the right, we get using the expressions obtained above for N, ω_0, N^1 :

$$\begin{aligned} 0 &= f(f^2)_r\Pi_q \left[-kB_1 \int^r du e^{ku} f(u) - B_2 e^{kr} + e^{kr}(B_1 g(r) + B_2) \right] \\ &= f(f^2)_r\Pi_q \left[-kB_1 \int^r du e^{ku} f(u) + e^{kr} B_1 g(r) \right] \\ &= f(f^2)_r\Pi_q \int^r du f^{-3}(u). \end{aligned} \quad (110)$$

where we integrated by parts to get the last equality. Since we have already seen that $B_1 = \dot{\Pi}_q$, the most general nontrivial solution is $B_1 = 0$, so that Π_q is a spatial and temporal constant.

To compute the shift vector N^1 , we solve (73). The solution contains an arbitrary function of time, but this in turn is required to be zero by (102).

We now change the spatial coordinate from x to r , so that $\dot{r} = 0$. We also change to a new time coordinate τ by $d\tau = B_2(t)dt$. Then all the equations of motion and constraints are satisfied by

$$f^2 = C_1 e^{-kr} + k\lambda^2(kr - 1)^2 + \Pi_q^2; \quad (111)$$

$$N(\tau, r) = f e^{kr}; \quad (112)$$

$$\omega_0(\tau, r) = -f f_r e^{kr}; \quad (113)$$

$$N^1(\tau, r) = -\Pi_q e^{kr}. \quad (114)$$

The metric is

$$ds^2 = -N^2 d\tau^2 + \frac{1}{f^2} (dr + f N^1 d\tau)^2. \quad (115)$$

The Ricci scalar and torsion of the above are, respectively:

$$R = -2k\lambda^2(r^2 + kr - 1) \quad (116)$$

$$T^0 = -kNq\Pi_q = -k e^{kr} (C_1 e^{-kr} + k\lambda^2(kr - 1)^2 + \Pi_q^2) \quad (117)$$

$$T^1 = Nq\Pi_q = \Pi_q e^{kr} \sqrt{C_1 e^{-kr} + k\lambda^2(kr - 1)^2 + \Pi_q^2} \quad (118)$$

Thus the general solution with torsion depends on two integration constants, C_1 and Π_q . Event horizons exist for negative C_1 .

VI. CONCLUSIONS

We have presented a study of DS/ADS/Poincare Yang-Mills gravity. As in general relativity, the theory is background independent, although this is done at the expense of reducing the symmetry group. We have shown that test ‘Higgs particles’ traverse geodesics with respect to the Lorentzian geometry determined by e_μ^i .

In two spacetime dimensions the action is a special case of the Katanaev-Volovich model. We completed the Hamiltonian analysis of the vacuum theory, confirming the existence of a generalized Birkhoff theorem: the solutions are static and parametrized by two parameters.

In addition one of us (JG) is working on Yang-Mills gravity in 4D with gauge group $SO(4,2)$, in collaboration with S. Rahmati and S. Seahra. In most work along these lines (see e.g. [4, 5]), the torsion is forced to be zero *ab initio*. We will relax this, and explore implications, especially for cosmology.

We close by mentioning that there is another possibility in principle allows the construction of an action that is invariant under the full gauge group. One can introduce two metrics: one is dynamical, and determined by the gauging e_μ^a of the generator J_a . The other metric, the background $g_{\mu\nu}$ is chosen in a way informed by the uniformization theorems in 2 and 3D. That is, given the topology, the manifold will admit a particular ‘round’ or homogeneous metric. The idea is to choose the background to be precisely that round geometry. This procedure is well defined

in two or in three dimensions, but there is a problem in four or higher dimensions, for which there is no known uniformization theorem. Recall that the 3D uniformization theorem was proved using the Ricci flow [12]. The latter exists in any dimension, and always converges to its fixed points, the homogeneous geometries. So one could require that the background geometry is such that the Ricci flow of the geometry determined by the frame fields and spin connection converges to it in the infinite limit of the flow parameter. However, given that we are really interested in working with YM type actions, it is more sensible to postulate that the consistency is provided by requiring that the Yang-Mills flow of the gauge potential A determined by the frame fields and spin connection flows to the background ‘round geometry’. In future work we will therefore consider an alternate theory wherein the background a metric which is the appropriate homogeneous geometry for some topology.

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- [1] P.K. Townsend *Small-scale structure of spacetime as the origin of the gravitational constant*, Phys. Rev. **D15**, 2795 (1977).
 - [2] H. Weyl, *Space-Time-Matter*, Methuen, London (1918).
 - [3] A sample, not exhaustive, list of papers on Yang-Mills gravity from the 70’s and 80’s: R. Utiyama, ‘Invariant theoretical interpretation of interaction’, Physical Review **101**,1597 (1956);doi:10.1103/PhysRev.101.1597; T. W. B. Kibble, ‘Lorentz invariance and the gravitational field’, J. Math. Phys. **2**,212 (1961); gravitational constant’, Phys. Rev. D **15**, 2795 (1977). S. W. MacDowell and F. Mansouri, ‘Unified geometric theory of gravity and supergravity’, Phys. Rev. Lett. **38** , 739742 (1977); K. Hayashi and T. Shirafuji, ‘Gravity from Poincaré gauge theory of fundamental interactions’, Prog. Theor. Phys. **64** , 866882 (1980); E. A. Ivanov and J. Niederle, ‘Gauge formulation of gravitational theories. I. The Poincaré, de Sitter, and conformal cases’, Phys. Rev. D **25**, 976 (1981).
 - [4] Again a sample of the literature in this area would include: J. T. Wheeler, ‘Auxiliary field in conformal gauge theory’, Phys. Rev. d **44**, 1769 (1991); de Sitter gravity’, Class. Quant. Grav. **24**, 4009 (2007). or on ‘conformal gravity’.
 - [5] H.-Y. Guo, et. al., ‘Snyder’s model-de Sitter special relativity duality and de Sitter gravity’, Class. Quant. Grav. **24**, 4009 (2007); C.-G. Huang, H.-Q. Zhang, H.-Y. Guo: *Cosmological solutions with torsion in a model of the de Sitter gauge theory of gravity*. JCAP **10** (2008) 010; C.-G. Huang, M.-S. Ma: On torsion-free vacuum solutions of the model of de Sitter gauge theory of gravity (II). Front. Phys. China, **4** (2009) 525529; C.-G. Huang, M.-S. Ma: *de Sitter spacetimes with torsion in the model of de Sitter gauge theory of gravity*. Phys. Rev. D **80** (2009) 084033; C.-G. Huang, M.-S. Ma: *A new solution with torsion in model of dS gauge theory of gravity*. Commun. Theor. Phys. **55** (2011) 6568.
 - [6] J. Gegenberg, A. C. Day, H. Liu and S. S. Seahra ‘An instability of hyperbolic space under the Yang-Mills flow’ Journal of Mathematical Physics, **55** , 042501 arXiv: 1210.0839 [hep-th].
 - [7] M. O. Katanaev and I. V. Volovich, String model with dynamical geometry and torsion, Phys. Lett. B **175** (1986) 413416. [arXiv:hep-th/0209014].
 - [8] P. Schaller and T. Strobl, “Canonical Quantization of Non-Einsteinian Gravity and the Problem of Time”, Class. Quant. Grav. **11** (1994) 331-346 [hep-th/9211054] ; Noriaki Ikeda and Ken-Iti Izawa, “Quantum Gravity with Dynamical Torsion in Two Dimensions” Prog. Theor. Phys. **89** (1993) 223-230; T. Strobl, “Comment on Gravity and the Poincaré Group”, Phys.Rev. D **48** (1993) 5029-5031 [arXiv:hep-th/9302041]; W. Kummer and D. J. Schwarz, General analytic solution of R^{*2} gravity with dynamical torsion in two-dimensions, Phys. Rev. D **45** (1992) 36283635. M. O. Katanaev, W. Kummer, and H. Liebl, Geometric interpretation and classification of global solutions in generalized dilaton gravity, Phys. Rev. D **53** (1996) 56095618 [gr-qc/9511009].
 - [9] R. Jackiw, in Quantum Theory of Gravity , edited by S. Christensen (Hilger, Bristol, 1984), p. 403; C. Teitelboim, in Quantum Theory of Gravity , edited by S. Christensen (Hilger, Bristol, 1984), p.327; M. Henneaux, Phys. Rev. Lett. **54** , 959 (1985). See also D. Louis-Martinez, J. Gegenberg and G. Kunstatter, Phys. Lett. B **321** , 193 (1994).
 - [10] E. Inönü, E.P. Wigner, *On the Contraction of Groups and Their Representations*, Proc. Nat. Acad. Sci. **39** (6): 51024 (1953).

- [11] O. Brodbeck and N. Straumann *A generalized Birkhoff theorem for the Einstein-Yang-Mills system*, J. Math. Phys. **34**, 2412-2423 (1993); T. A. Oliynyk and H. P. Kunzle, *On all possible static spherically symmetric EYM solitons and black holes*, Class. Quant. Grav. **19**, 457 (2002), [arXiv:gr-qc/0109075].
- [12] The original geometrization conjecture of W.P. Thurston was proved by J. Hamilton and G. Perelman. For references, see J. W. Morgan. *Recent progress on the Poincar conjecture and the classification of 3-manifolds*, Bulletin Amer. Math. Soc. 42 (2005) no. 1, 57-78.